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A statistical theory of turbulent relative dispersion

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The laws governing the spread of a cluster of particles in homogeneous isotropic turbulence are derived using a theoretical approach based on inertial subrange scaling and statistical diffusion theory. The equations for the mean square dispersion of a puff admit an analytical solution in the inertial subrange and at large scales. The solution is consistent with Taylor's theory of absolute dispersion. An analytical derivation of the Richardson–Obukhov constant of relative dispersion is presented. A time scale for relative dispersion is identified, as well as relations between Lagrangian and Eulerian structure functions. The results are extended to turbulence at finite Reynolds number. A closure assumption for the relative kinetic energy, based on Taylor's theory, is presented. Comparisons with direct numerical simulations and laboratory experiments are reported.

1. Introduction

In 1921, G. I. Taylor developed a fundamental theory to predict the time evolution of the variance of the spatial distribution of marked fluid particles about a fixed origin, in statistically homogeneous turbulent flow. The spreading of particles with respect to a fixed origin is known as absolute dispersion. Taylor's (1921) dispersion theory is purely kinematic – no dynamic balance of forces is invoked – and is based on the concurrent application of basic mechanical and statistical definitions, such as velocity and velocity autocovariance of fluid particles. The resulting law of dispersion is an exact analytical tool whose elegance and simplicity are a consequence of the kinematic nature of the approach. The time-averaged spread of a continuous release, or of a series of instantaneous releases, of a passive scalar with respect to a fixed point can be determined if the Lagrangian velocity autocovariance is known. In stationary conditions, realistic parameterized forms of the velocity autocovariance are relatively easy to construct and test because of its independence of time.

The spreading of marked fluid particles around their instantaneous centre of mass is known as relative dispersion. The application of Taylor's theory to the derivation of a parallel expression for the relative dispersion variance has not been widely explored because of a number of difficulties. Dispersion statistics (both absolute and relative) are in effect generated by double averages – over the particles in each realization of the cluster and over the ensemble of realizations. If the flow is stationary, a single particle in each realization is sufficient to evaluate absolute dispersion statistics, and the double average reduces to a single average. This simplification does not apply to relative dispersion: at any given instant in time, the basic variable used to calculate relative dispersion statistics is the distance of a particle from the instantaneous centre of mass, which is determined by at least two particles in each realization. Also, the autocovariance of relative velocity is a function of time, as well as of the time lag, even in stationary turbulence. Because of the Lagrangian character of the variables, and of the difficulties inherent in the time dependence of the process, reference experimental data are scarce and unrepresentative, and parameterization of the relative velocity autocorrelation function is impractical. A predictive closed theory for relative dispersion equivalent to Taylor's is not available.

From the analysis of various types of atmospheric dispersion data, Richardson (1926) recorded, in terms of eddy diffusivity, the $\mathrm{d}\langle \boldsymbol{r}^2\rangle/\mathrm{d}t \propto (\sqrt{\langle \boldsymbol{r}^2\rangle})^{4/3}$ power law, where $\langle \boldsymbol{r}^2\rangle$ is the mean square separation between particles, which for a puff is simply twice the mean square separation from the instantaneous centre of mass. Although the index 4/3 was determined by Richardson on an empirical basis, and the range of experimental values in his dataset exceeded the inertial subrange external length scale, it is generally recognized that this choice 'indicates his faith in the existence of a universal physical law of sufficiently simple form' (Monin & Yaglom 1975, p. 557) and was therefore, at least intuitively, based on dimensional arguments. In fact, Taylor (1959) goes as far as to suggest that Richardson 'had the idea that the index was determined by something connected with the way energy was handed down from larger to smaller and smaller eddies.' Following an application of Kolmogorov's inertial subrange theory to relative dispersion, Obukhov (1941) proposed the dependence of $\mathrm{d}\langle \boldsymbol{r}^2\rangle/\mathrm{d}t$ on the mean dissipation of kinetic energy ε which led to the classical relation $\langle \boldsymbol{r}^2\rangle = C_r \varepsilon t^3$, where C_r is the Richardson–Obukhov constant (see also the independent derivation by Batchelor 1950). Although the dimensional argument based on Kolmogorov's theory has been somewhat questioned (Lin & Reid 1963, p. 510), the validity of Obukhov's approach is today widely accepted (Batchelor & Townsend 1956, p. 386; Sawford 2001).

Nevertheless, the value of the universal proportionality coefficient C_r remains elusive.

Several factors contributed to the substantial uncertainty in the estimate of C_r through atmospheric measurements, laboratory experiments, and numerical simulations. Experimental evidence of Richardson's law (in the original 4/3 form or in Obukhov's t^3 form) is controversial. Monin & Yaglom (1975, pp. 557, 558 and 565) report at least thirteen papers describing experiments of two-dimensional diffusion on sea or lake surfaces in support of the 4/3 law, and at least seven in support of the t^3 law; the experimental support to the 4/3 or t^3 law is never questioned, and the analysis is rather focused on explaining the discrepancies found in the constants of proportionality. Other experiments are reported by Gifford (1977), who plots data which seem to display a t^3 behaviour, and concludes that '[s]everal more pieces of data could perhaps be added, but these are ample to support the following conclusions. For puffs or plumes released in the boundary layer there is, after the disappearance of initial source size effects, a $t^{3/2}$ spreading regime extending to somewhere between 1000 and 3000 s, ...'. On the contrary, Sawford (2001) states that '[w]ith perhaps the exception of the data of Frenkiel & Katz (1956) ..., there are no convincing demonstrations of the t^3 law, despite more recent attempts to analyse atmospheric dispersion data in this light.' Also, recent laboratory experiments hint at the possibility that the t^3 law may not hold even at medium to high Reynolds numbers unless the initial particle separation is significantly small (Bourgoin *et al.* 2006).

Indirect empirical estimates of relative dispersion from scalar variances are impractical because of the need to make additional assumptions on the internal scalar probability density function, and also because the observed scalar fluctuations are affected by sampling time and intermittency effects (e.g. Wilson 1995). This contrasts with the experimental measurements of absolute dispersion, where the particle spread can be exactly inferred from the simple observation of mean scalar fields. Therefore, measurements of mean square relative separation must be performed by direct detection of the simultaneous trajectories of clusters (or just pairs) of particles. Although these measurements have been accurately performed in laboratory flow at low Reynolds number, they still represent a challenging task in atmospheric flow.

Uncertainties affecting the measurements of C_r in the atmosphere originate from the imperfect homogeneity of turbulence caused by atmospheric stability effects, statistical errors due to the limited sampling time used in order to collect data under the same conditions, natural atmospheric variability and the presence of large-scale structures.

On the other hand, high-Reynolds-number turbulence necessary for the development of a well-defined clearly detectable inertial subrange spanning a few decades is difficult to achieve in the controlled conditions of a water tank or a wind tunnel. In addition, we will show that even for Reynolds numbers high enough to produce inertial subrange scaling, there is a Reynolds-number dependence of C_r that must be accounted for in experimental measurements. This suggests that there is a range of Reynolds numbers where the universal scaling properties of some, but not all, turbulence statistics are still preserved, and where universality does not apply to the proportionality constants. Ott & Mann's (2000) laboratory experiments at low Reynolds number (the Taylor-scale Reynolds number Re_λ was about 90 to 100, corresponding to $Re \sim 600$) indicate that inertial subrange scaling still holds for the structure function of the second order, but does not apply to that of the third order. Also, they observe t^3 dependence of $\langle \boldsymbol{r}^2 \rangle$. Evidence of the non-universal character of inertial subrange constants is also apparent in other studies (Sawford 1991; Heinz 2003, p. 104).

Inconsistent estimates of C_r between different experiments are also ascribable to the difficulty of performing simultaneous measurements of ε and $\langle \boldsymbol{r}^2 \rangle$. Usually, approximated estimates of ε are employed for the specific flows where relative dispersion is measured. For example, Ott & Mann (2000) performed an indirect estimate of ε based on the second-order Eulerian velocity structure function with the Kolmogorov constant $C_k = 2$.

Numerical simulations of relative separation have been performed essentially using five basic techniques: direct numerical simulations (DNSs); Lagrangian stochastic two-particle models; kinematic simulations; the eddy-damped quasi-normal Markovian (EDQNM) approximation; and the Lagrangian-history direct-interaction (LHDI) theory. These techniques have in general provided results that are inconsistent with each other.

DNS results seem to be reasonably consistent with each other as well as at least one laboratory experiment (Ott & Mann 2000). However, DNSs of turbulent flow are currently possible only at moderate Reynolds numbers, and are not exempt from statistical errors owing to the limited size of the ensemble, and from numerical approximations generated by the forcing technique used and by the limitations inherent in the discretization scheme.

Lagrangian stochastic two-particle models have the advantage of providing results independent of ε, as all calculations can be carried out on scaled equations. Most Lagrangian stochastic models using the acceleration drift term quadratic in the velocity (e.g. Thomson 1990; Borgas & Sawford 1994; Franzese & Borgas 2002) provide C_r consistently higher than DNS and laboratory estimates even accounting for Reynolds-number effects ($1 \lesssim C_r \lesssim 2$); linear drift term models (e.g. Novikov 1989; Pedrizzetti & Novikov 1994; Heppe 1998) typically give smaller values ($0.1 \lesssim C_r \lesssim 0.4$). The discrepancy may be due to difficulties accounting for the non-Gaussian statistics of turbulent velocity differences (Franzese & Borgas 2002), viscosity effects at small separations (Heppe 1998; Borgas & Yeung 2004), sensitivity to the Lagrangian time scale and to the Eulerian velocity structure functions.

Kinematic simulations (e.g. Fung *et al.* 1992; Elliott & Majda 1996; Flohr & Vassilicos 2000) provide much lower estimates of C_r, of the order 0.1. However, Thomson & Devenish (2005) cast doubt on the ability of kinematic simulations to reproduce t^3 scaling correctly in the inertial subrange.

A revised application of the EDQNM approximation (Larchevêque & Lesieur 1981) by Thomson (1996) provided $C_r = 1.4$. A much higher value ($C_r = 5.5$) was obtained by Ott & Mann (2000) from an application of the LHDI theory (Kraichnan 1966) after correcting a minor error in the original calculation (which had already been remarked on by Thomson 1996).

Several theoretical treatments proposed in the early 1960s were based on the assumption that the accelerations of the fluid particles at large Reynolds number are uncorrelated. They include an original application of the statistical diffusion theory by Lin (1960), the random force method developed by Novikov (1963), and relations between velocity and position structure functions obtained by Ivanov & Stratonovich (1963). All of the above provide the relation $C_r = 2C_o$, where C_o is the Lagrangian velocity structure function constant. Borgas & Sawford (1991) have shown that accounting for the two-particle acceleration covariance provides for a value of C_r smaller than $2C_o$ by an undetermined quantity which depends on the structure of the two-particle acceleration covariance. Mikkelsen, Larsen & Pécseli (1987) related the growth rate of a puff to common one-dimensional velocity spectra, under the assumptions that the displacements of a particle in the frame of reference moving with the centre of mass are uncorrelated; and that a two-particle velocity covariance can be estimated as the product of a two-point (Eulerian) space-correlation and a single-particle (Lagrangian) autocorrelation. (Batchelor 1952 expressed criticism of the latter approximation.) Mikkelsen *et al.* (1987) estimated $C_r = 3.2$ from a relation equivalent to $C_r \propto C_k^{3/2}$, where C_k is the constant in the second-order Eulerian longitudinal velocity structure function.

We derive a differential equation for the mean square particle separation from the centre of mass of a puff in one dimension, $\langle y_r^2 \rangle$, which is equal to $\frac{1}{6}\langle \boldsymbol{r}^2 \rangle$. The equation can be solved numerically, and has an analytical solution in the inertial subrange and at large scales. The derivation of $\langle y_r^2 \rangle$ is based on a statistical diffusion theory of relative dispersion, and on the inertial subrange scaling form of the turbulent kinetic energy of separation.

The plan of the paper is as follows. In the first part of § 2, we briefly describe the notation adopted and the types of averages that will be used thereafter. We then derive some statistical properties for clusters of particles, including the relation between mean square distance between the particles $\langle \boldsymbol{r}^2 \rangle$ and mean square distance of the particles from the centre of mass along an arbitrary direction $\langle y_r^2 \rangle$.

The governing equation for $\langle y_r^2 \rangle$ is derived in § 3, along with the definition of the decorrelation time scale for relative dispersion, which is the time integral of the relative velocity autocorrelation function. The relative dispersion time scale, as opposed to the autocorrelation function, is shown to be the fundamental quantity in the calculation of $\langle y_r^2 \rangle$.

Relative dispersion in the inertial subrange is investigated in § 4: we present a derivation of the solution in the inertial subrange; the analytical expression of C_r as a function of C_o; the relation between C_o and C_k; and the extension of the results to lower-Reynolds-number turbulence. The large-scale solution is presented in § 5, and the solution in the vicinity of a finite source is discussed in § 6.

A closure for the relative turbulent kinetic energy that appears in the governing equation is determined in § 7, where a large-eddy length scale is defined based on the statistical theory of absolute dispersion.

In § 8, several predictions are compared to what appear, at present, to be reliable datasets accounting for Reynolds-number dependence, namely DNS and laboratory observations.

2. Statistical properties of a cluster of particles

The velocity of a particle along its trajectory $\boldsymbol{y}(t)$ will be indicated by $\boldsymbol{v}(t) = \mathrm{d}\boldsymbol{y}/\mathrm{d}t$, the components of $\boldsymbol{y}$ and $\boldsymbol{v}$ along an arbitrary y-axis are denoted by y and v, respectively. Two types of averages will be used: over a cluster of particles in one single realization of the velocity field; and over a statistical ensemble of realizations of the turbulent flow.

Overbars represent averaging over a cluster of N particles in one single realization of the velocity field:

$$\overline{y^n}(t) = \frac{1}{N}\sum_{i=1}^{N} y_i^n, \qquad \overline{v^n}(t) = \frac{1}{N}\sum_{i=1}^{N} v_i^n, \tag{2.1}$$

are the moments of position and velocity of the particles (in one realization of the flow), where y_i and v_i denote position and velocity of the ith particle along an arbitrary y-axis.

Angle brackets indicate averaging over an ensemble of realizations of the flow field: the moments of position and velocity of the particles are defined as

$$\langle y^n(t)\rangle = \int y^n p(y)\,\mathrm{d}y, \qquad \langle v^n(t)\rangle = \int v^n p(v)\,\mathrm{d}v, \tag{2.2}$$

where p indicates a probability density function.

Because the averages are unconditional, the following operators are equivalent: $\overline{\langle\,\cdot\,\rangle} \equiv \langle\overline{\,\cdot\,}\rangle \equiv \langle\,\cdot\,\rangle$, and hereinafter only simple angle brackets will be used to indicate double averages. The use of unconditional averages implies that the averages are taken long enough after release that the effects of initial conditions have disappeared.

For simplicity, we assume turbulence with no mean flow. However, the equations derived in this paper are also valid for constant mean flow, if the variables are expressed in an inertial coordinate system moving with the mean velocity. Note that even in a stationary, homogeneous and isotropic turbulence with no mean flow and a spherical source centred on the origin, $\overline{\boldsymbol{v}}(t) \neq 0$ and $\overline{\boldsymbol{y}}(t) \neq 0$ for some t, while $\langle \boldsymbol{v}(t)\rangle = 0$ and $\langle \boldsymbol{y}(t)\rangle = 0$.

Position and velocity of a particle can be decomposed as

$$\boldsymbol{y} = \overline{\boldsymbol{y}} + \boldsymbol{y}_r, \qquad \boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}_r, \tag{2.3}$$

which define $\boldsymbol{y}_r$ and $\boldsymbol{v}_r$ as position and velocity of a particle relative to the centre of mass of the cluster. From (2.3), the mean square distance of the particles from the origin and the mean square velocity can be partitioned as follows:

$$\langle \boldsymbol{y}^2 \rangle = \langle \overline{\boldsymbol{y}}^2 \rangle + \langle \boldsymbol{y}_r^2 \rangle, \qquad \langle \boldsymbol{v}^2 \rangle = \langle \overline{\boldsymbol{v}}^2 \rangle + \langle \boldsymbol{v}_r^2 \rangle. \tag{2.4}$$

Combining (2.1) and (2.4), we derive

$$\langle \overline{\boldsymbol{y}}^2 \rangle = \frac{1}{N} \langle \boldsymbol{y}^2 \rangle + \left(1 - \frac{1}{N}\right) \langle \boldsymbol{y}_i \boldsymbol{y}_j \rangle, \tag{2.5}$$

$$\langle \boldsymbol{y}_r^2 \rangle = \left(1 - \frac{1}{N}\right) \left(\langle \boldsymbol{y}^2 \rangle - \langle \boldsymbol{y}_i \boldsymbol{y}_j \rangle\right), \tag{2.6}$$

for any $i, j \leqslant N$ with $i \neq j$. Analogous relations can be derived for $\langle \overline{\boldsymbol{v}}^2 \rangle$ and $\langle \boldsymbol{v}_r^2 \rangle$.

The vector $\boldsymbol{r}$ determines the relative position of two particles, i.e. $\boldsymbol{r} = \boldsymbol{y}_i - \boldsymbol{y}_j$. The mean square distance between the particles over an ensemble of realizations is defined as

$$\langle \boldsymbol{r}^2 \rangle = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1}^{N} \langle (\boldsymbol{y}_i - \boldsymbol{y}_j)^2 \rangle; \tag{2.7}$$

from (2.7) and (2.6) we obtain the fundamental expression

$$\langle \boldsymbol{r}^2 \rangle = \frac{2N}{N-1} \langle \boldsymbol{y}_r^2 \rangle, \tag{2.8}$$

which shows that the mean square distance of the particles of a cluster from the centre of mass $\langle \boldsymbol{y}_r^2 \rangle$, which is a measure of the size of the cluster, depends on the number of particles in the cluster, as well as on the ensemble mean square separation between the particles $\langle \boldsymbol{r}^2 \rangle$. Therefore, for clusters consisting of two particles, $\langle \boldsymbol{r}^2 \rangle = 4 \langle \boldsymbol{y}_r^2 \rangle$, whereas for a continuous distribution of particles (or $N \to \infty$), we recover the well-known relation $\langle \boldsymbol{r}^2 \rangle = 2 \langle \boldsymbol{y}_r^2 \rangle$ (Batchelor 1952).

Because of isotropy, the mean square of a vector is three times the mean square of one of its components: $\langle \boldsymbol{y}_r^2 \rangle = 3 \langle y_r^2 \rangle$. As a consequence,

$$\langle \boldsymbol{r}^2 \rangle = \frac{6N}{N-1} \langle y_r^2 \rangle. \tag{2.9}$$

In this paper, we will derive an expression for $\langle y_r^2 \rangle$, assuming a continuous distribution of particles in the cluster. The Richardson–Obukhov law for $\langle \boldsymbol{r}^2 \rangle$, and the Richardson–Obukhov constant C_r, follow immediately from (2.9). Figure 1 shows a three-dimensional Gaussian distribution of particles, obtained as a superposition of several realizations to represent ensemble statistics. The two grey spheres represent the iso-probability surfaces corresponding to $|\boldsymbol{y}_r| = \sqrt{\langle \boldsymbol{y}_r^2 \rangle}$ (internal sphere) and $|\boldsymbol{y}_r| = \sqrt{\langle \boldsymbol{r}^2 \rangle}$ (external sphere). The values of $\sqrt{\langle \boldsymbol{y}_r^2 \rangle} = \sqrt{3 \langle y_r^2 \rangle}$ and $\sqrt{\langle \boldsymbol{r}^2 \rangle} = \sqrt{6 \langle y_r^2 \rangle}$ are also reported onto the y_r-axis.

The letter u indicates the Eulerian counterpart of the Lagrangian variable v. Because we consider homogeneous turbulence, we will often refer to $\langle v^2 \rangle$ as 'turbulent kinetic energy', although $\langle v^2 \rangle$ is in fact rigorously proportional to the turbulent kinetic energy per unit mass k, namely $\langle v^2 \rangle = \langle u^2 \rangle = 2k/3$ (Lumley 1961).

In the next section, the governing equations for the evolution of $\langle y_r^2 \rangle$ are derived. The equations could be expressed directly in terms of $\boldsymbol{r}$ or $\boldsymbol{y}_r$. However, the present derivation is more intuitive, and establishes a direct parallel with Taylor's theory, which was originally derived for one-dimensional dispersion. Extension of the results to three dimensions is straightforward.

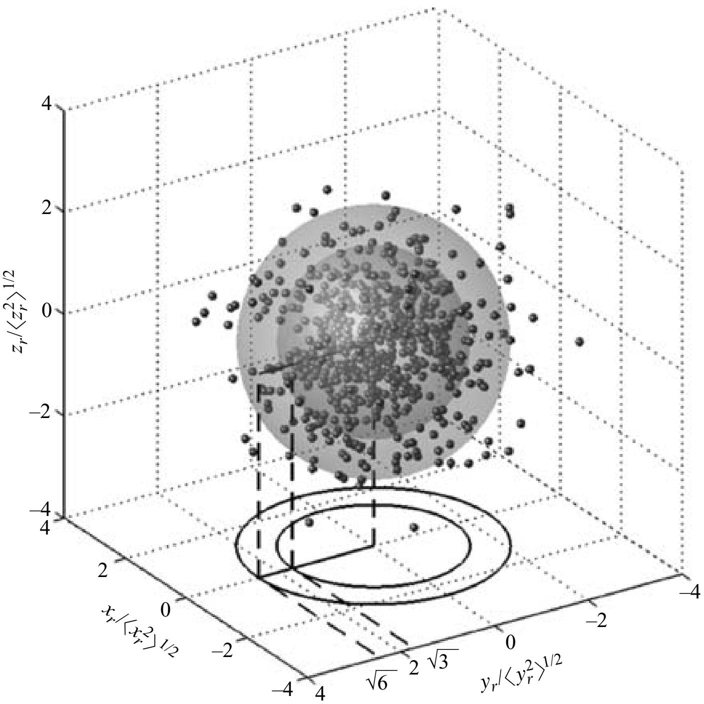

FIGURE 1. Three-dimensional Gaussian distribution of particles. The two grey spheres represent the iso-probability surfaces corresponding to $|\boldsymbol{y}_r| = \sqrt{\langle \boldsymbol{y}_r^2\rangle}$ (internal sphere) and $|\boldsymbol{y}_r| = \sqrt{\langle \boldsymbol{r}^2\rangle}$ (external sphere). The values of $\sqrt{\langle \boldsymbol{y}_r^2\rangle} = \sqrt{3\langle y_r^2\rangle}$ and $\sqrt{\langle \boldsymbol{r}^2\rangle} = \sqrt{6\langle y_r^2\rangle}$ are also reported onto the y_r-axis.

3. Governing equations

In this section, we derive the evolution equation for the mean square distance of the particles of a cluster from the instantaneous centre of mass over an ensemble of realizations, $\langle y_r^2\rangle$. The solutions in the inertial subrange and at large scales will be presented in §§4 and 5, respectively.

The variable $\langle y_r^2(t)\rangle$ satisfies the differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle y_r^2\right\rangle = 2\left\langle y_r \frac{\mathrm{d}y_r}{\mathrm{d}t}\right\rangle = 2\int_0^t \langle v_r(t)v_r(t')\rangle\,\mathrm{d}t', \tag{3.1}$$

where the release time $t_o = 0$ for simplicity. Operating the change of variable $\tau = t - t'$, (3.1) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle y_r^2\right\rangle = 2\int_0^t \langle v_r(t)v_r(t-\tau)\rangle\,\mathrm{d}\tau = 2\left\langle v_r^2(t)\right\rangle \int_0^t R_r(t,\tau)\,\mathrm{d}\tau, \tag{3.2}$$

where the autocorrelation coefficient R_r of Lagrangian relative velocity v_r was defined as

$$R_r(t,\tau) = \frac{\langle v_r(t)v_r(t-\tau)\rangle}{\langle v_r^2(t)\rangle}, \tag{3.3}$$

with $0 \leqslant \tau \leqslant t$. R_r is a function of the two variables t and τ because the relative velocity v_r is a non-stationary random function. This is a consequence of the range of eddy sizes contributing to v_r increasing with time as long as the particles have correlated motions; relative dispersion is an accelerating process (Batchelor 1952). Equation (3.2) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle y_r^2\right\rangle = 2\left\langle v_r^2\right\rangle T_r, \tag{3.4}$$

where

$$T_r(t) = \int_0^t R_r(t,\tau)\,\mathrm{d}\tau \tag{3.5}$$

is a time-dependent Lagrangian decorrelation time scale for turbulent relative dispersion. According to Csanady (1973, p. 92), '[t]he Lagrangian time-scale T_r is characteristic of all those eddies contributing to relative velocities, which we have seen to be mainly those comparable in size to typical particle separation in cloud (i.e. also to cloud size). As the cloud grows, larger eddies begin to contribute, which, generally speaking, have longer 'lifetimes', so that the Lagrangian time-scale T_r slowly increases. So does, of course, the mean-square relative velocity.'

We will obtain T_r following a procedure well established in the context of one-particle autocorrelations (Tennekes 1979): we (i) determine the inertial subrange expression of R_r; (ii) assume the subrange expression of R_r to be the small-time expansion of an asymptotically decaying function valid within and beyond inertial subrange scales; and (iii) estimate the Lagrangian time scale T_r from this function.

As a result, we will obtain a differential equation for $\langle y_r^2\rangle$, which is valid within and beyond inertial subrange scales.

The remainder of this section is structured as follows: the relative velocity covariance $\langle v_r(t)v_r(t-\tau)\rangle$ which appears in (3.3) is determined in § 3.1; the autocorrelation R_r, the Lagrangian relative time scale T_r, the mean square relative velocity $\langle v_r^2\rangle$ and the evolution equation for $\langle y_r^2\rangle$ are given in § 3.2.

3.1. *Lagrangian relative velocity autocovariance and structure function*

The velocity autocovariance $\langle v_r(t)v_r(t-\tau)\rangle$ can be expressed as

$$\langle v_r(t)v_r(t-\tau)\rangle = \tfrac{1}{2}\left[\left\langle v_r^2(t)\right\rangle + \left\langle v_r^2(t-\tau)\right\rangle - \langle(\Delta_\tau v_r)^2\rangle\right], \tag{3.6}$$

where $\Delta_\tau v_r = v_r(t) - v_r(t-\tau)$, and $\langle(\Delta_\tau v_r)^2\rangle$ is a Lagrangian relative velocity structure function. From (2.4), we can write

$$\langle(\Delta_\tau v_r)^2\rangle = \langle(\Delta_\tau v)^2\rangle - \langle(\Delta_\tau \overline{v})^2\rangle \tag{3.7}$$

In each realization of a cluster composed by N particles, the velocity time difference for the centre of mass is defined as

$$\Delta_\tau \overline{v} = \frac{1}{N}\sum_{i=1}^{N}\Delta_\tau v_i \tag{3.8}$$

and, after squaring and averaging over the ensemble of realizations,

$$\langle(\Delta_\tau \overline{v})^2\rangle = \frac{1}{N}\langle(\Delta_\tau v)^2\rangle + \left(1-\frac{1}{N}\right)\langle\Delta_\tau v_i\,\Delta_\tau v_j\rangle \tag{3.9}$$

for any $i, j \leqslant N$ with $i \neq j$. Therefore,

$$\langle(\Delta_\tau v_r)^2\rangle = \left(1-\frac{1}{N}\right)[\langle(\Delta_\tau v)^2\rangle - \langle\Delta_\tau v_i\,\Delta_\tau v_j\rangle]. \tag{3.10}$$

Our goal is to determine the expression of R_r in the inertial subrange, and then generalize it to larger scales. According to the Kolmogorov similarity hypotheses for locally isotropic turbulence,

$$\langle (\Delta_\tau v)^2 \rangle = C_o \varepsilon \tau \tag{3.11}$$

for $1 \ll \tau/\tau_\eta \ll Re^{1/2}$, where the Kolmogorov time scale $\tau_\eta = (\nu/\varepsilon)^{1/2}$, ν is the viscosity, ε is the mean dissipation rate of turbulent kinetic energy per unit mass, and C_o is a universal constant. Since, in the inertial subrange, the acceleration correlation function is short-ranged both in space and time, the acceleration of a particle is only weakly correlated with that of any other particle (Monin & Yaglom 1975, p. 546; Thomson 1990). Therefore, the velocity time increments of different particles are weakly correlated with each other, and $\langle \Delta_\tau v_i \Delta_\tau v_j \rangle$ is generally assumed to be negligible compared to $\langle (\Delta_\tau v)^2 \rangle$ (Kurbanmuradov & Sabelfeld 1995; Kurbanmuradov *et al.* 2001; Franzese & Borgas 2002; Borgas & Yeung 2004).

Substituting (3.11) into (3.10), and neglecting $\langle \Delta_\tau v_i \Delta_\tau v_j \rangle$, we have

$$\langle (\Delta_\tau v_r)^2 \rangle \approx \left(1 - \frac{1}{N}\right) C_o \varepsilon \tau, \tag{3.12}$$

which gives, for a large number of particles N,

$$\langle (\Delta_\tau v_r)^2 \rangle \approx C_o \varepsilon \tau. \tag{3.13}$$

Equation (3.13), together with (3.7), shows that the velocity structure function for the centre of mass of a puff is negligible compared to the velocity structure function of a particle. Note that for the case of a pair of particles ($N = 2$), since the relative velocity between two particles $v_i - v_j = 2v_r$, (3.12) gives the two-particle Lagrangian longitudinal structure function commonly used in quasi-one-dimensional relative dispersion stochastic models, i.e. $\langle [\Delta_\tau (v_i - v_j)]^2 \rangle \approx 2C_o \varepsilon \tau$ (Kurbanmuradov & Sabelfeld 1995; Franzese & Borgas 2002; Borgas & Yeung 2004).

3.2. *Autocorrelation function, Lagrangian time scale and evolution equation*

Substituting (3.6) and (3.13) into (3.3), $R_r(t, \tau)$ is expressed as

$$R_r(t,\tau) = \frac{\langle v_r(t) v_r(t-\tau) \rangle}{\langle v_r^2(t) \rangle} = 1 - \frac{C_o \varepsilon}{2\langle v_r^2(t) \rangle}\tau - \frac{\langle v_r^2(t) \rangle - \langle v_r^2(t-\tau) \rangle}{2\langle v_r^2(t) \rangle}. \tag{3.14}$$

According to (2.4), the turbulent kinetic energy $\langle v^2 \rangle$ can be decomposed as $\langle v^2 \rangle = \langle \overline{v}^2 \rangle + \langle v_r^2 \rangle$. The term $\langle \overline{v}^2 \rangle$ is the energy associated with the motion of the centre of mass, which is caused by the eddies larger than a characteristic size $\mathscr{D}$ of the cluster. The term $\langle v_r^2 \rangle$ is therefore the energy contained in the remaining part of the spectrum, namely the energy of all eddies of size smaller or comparable to $\mathscr{D}$. According to inertial subrange scaling $\langle v_r^2 \rangle \propto (\varepsilon \mathscr{D})^{2/3} \propto t$. Therefore, (3.14) becomes:

$$R_r(t,\tau) = 1 - \frac{C_o \varepsilon}{2\langle v_r^2 \rangle}\tau - \frac{1}{2t}\tau, \tag{3.15}$$

which is exact within the limits of the inertial subrange.

In order to obtain a more general expression, valid beyond inertial subrange scales, we will use the same logical steps commonly taken to define the standard autocorrelation function for absolute velocity $R(\tau)$ from its inertial range expression and from the Lagrangian time scale $T_L = 2\langle v^2 \rangle/(C_o \varepsilon)$ (e.g. Corrsin 1963; Tennekes 1979). To this extent, we note that (3.15) may be interpreted as a Taylor expansion at small τ of some function valid beyond inertial subrange time scales. Equation (3.2) shows that the exact form of $R_r(t, \tau)$ is relatively unimportant in the calculation of $\langle y_r^2 \rangle$, whereas its integral $T_r = \int_0^t R_r \mathrm{d}\tau$ (3.5) determines the solution directly.

For simplicity, we assume a function of the exponential-decay type, i.e. $R_r(t,\tau) = \exp(-\tau/T_r)$, which satisfies (3.5) provided that $R_r(t,\tau) \approx 0$ for $\tau = t$. This assumption is justified by theoretical considerations (Csanady 1973, p. 92), and is very well supported by experiments and DNS (Jullien, Paret & Tabeling 1999; Goto & Vassilicos 2004; Biferale *et al.* 2005).

Matching the small-time first-order Taylor series expansion of $\exp(-\tau/T_r)$ with the inertial subrange form of $R_r(t,\tau)$ given in (3.15), we obtain:

$$1 - \frac{C_o\varepsilon}{2\langle v_r^2\rangle}\tau - \frac{1}{2t}\tau = 1 - \frac{\tau}{T_r} + O(\tau^2), \tag{3.16}$$

which provides:

$$T_r = \left(\frac{C_o\varepsilon}{2\langle v_r^2\rangle} + \frac{1}{2t}\right)^{-1}. \tag{3.17}$$

This expression of T_r is valid both in the inertial subrange and at larger scales, but is not rigorous in the transition regime at scales comparable to the turbulence scale. However, the second term in parentheses vanishes at large t, defining a transition between different regimes without the need to impose an artificial interpolation to adjust T_r. It can be remarked that because at large dispersion times $\langle v_r^2\rangle$ tends to $\langle v^2\rangle$, T_r tends to T_L. As a consequence, $R_r(t,\tau)$ tends to become independent of t and equal to $R(\tau)$.

Equation (3.4) becomes, after substituting T_r from (3.17):

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle y_r^2\rangle = \frac{4\langle v_r^2\rangle^2 t}{\langle v_r^2\rangle + C_o\varepsilon t}, \tag{3.18}$$

which, by virtue of (3.17), is valid in the inertial subrange and at large scales, but may inaccurately describe the transition regime. In the next section, the inertial subrange solution of (3.18) is derived, as well as additional relations between $\langle v_r^2\rangle$ and the Eulerian second-order velocity structure function, and an analysis of Reynolds-number effects, which will be used in the comparison with experiments and DNS. The large-scale solution of (3.18) is presented in § 5.

4. Inertial subrange

4.1. *Solution in the inertial subrange*

Before obtaining the solution to (3.18) in the inertial subrange, we must derive $\langle v_r^2\rangle$ in terms of known quantities. Because $\langle y_r^2\rangle$ represents the square of a characteristic size of the puff, according to inertial subrange scaling $\langle y_r^2\rangle \propto \langle v_r^2\rangle^3/\varepsilon^2$. If we indicate by $\mathscr{L}_L$ a length scale of the energy-containing eddies, we also have: $\mathscr{L}_L^2 \propto \langle v^2\rangle^3/\varepsilon^2$. Therefore, $\langle v_r^2\rangle$ can be obtained from the ratio $\langle y_r^2\rangle/\mathscr{L}_L^2$; it is important to note that other variables can equally be chosen instead of $\sqrt{\langle y_r^2\rangle}$ to represent the characteristic size of the cloud, as long as $\mathscr{L}_L$ is consistent with this choice. A closure assumption for $\mathscr{L}_L$ is presented in § 7.

We can now write:

$$\langle v_r^2\rangle = \langle v^2\rangle\left(\frac{\langle y_r^2\rangle}{\mathscr{L}_L^2}\right)^{1/3}, \tag{4.1}$$

which is valid in the inertial subrange, namely for $\eta^2 \ll \langle y_r^2 \rangle \ll \mathscr{L}_L^2$. (The evolution of $\langle v_r^2 \rangle$ for $\langle y_r^2 \rangle \simeq \mathscr{L}_L^2$ should, in principle, be determined by a different law, which should include the large-time limit $\langle v_r^2 \rangle = \langle v^2 \rangle$ for $\langle y_r^2 \rangle \gg \mathscr{L}_L^2$.)

Note that, by definition, the proportionality $\mathscr{L}_L^2 \propto \langle v^2 \rangle^3/\varepsilon^2$ can be expressed as

$$\mathscr{L}_L^2 = C_L \langle v^2 \rangle T_L^2, \tag{4.2}$$

where $T_L = 2\langle v^2 \rangle/(C_o \varepsilon)$ is the one-particle Lagrangian time scale of turbulence, and the constant C_L will be determined by the closure assumption in § 7.

Although (3.18) in the inertial subrange is an implicit equation as $\langle v_r^2 \rangle$ depends on $\langle y_r^2 \rangle$, it can be solved analytically, and admits the following solution:

$$\langle y_r^2 \rangle = C_y \varepsilon t^3 \tag{4.3}$$

with

$$C_y = \alpha C_o \tag{4.4}$$

and the constant

$$\alpha = \frac{1}{2} \frac{\mathscr{L}_L^2}{\langle v^2 \rangle T_L^2} \left(\sqrt{1 + \frac{4}{3} \frac{\langle v^2 \rangle T_L^2}{\mathscr{L}_L^2}} - 1 \right)^3 = \frac{C_L}{2} \left(\sqrt{1 + \frac{4}{3C_L}} - 1 \right)^3. \tag{4.5}$$

According to (2.9), the relation between $\langle y_r^2 \rangle$, which is the one-dimensional mean square distance from the centre of mass, and Richardson's $\langle \boldsymbol{r}^2 \rangle$, which is the mean square distance between particles, is simply $\langle \boldsymbol{r}^2 \rangle = 6 \langle y_r^2 \rangle$. Therefore, the relation between C_y and the more familiar Richardson–Obukhov two-particle relative dispersion constant C_r which appears in the equation $\langle \boldsymbol{r}^2 \rangle = C_r \varepsilon t^3$ is given by $C_r = 6C_y$, which implies:

$$C_r = 6\alpha C_o. \tag{4.6}$$

The proportionality between C_r and C_o is physically sound. C_o is defined by the structure function (3.11), and is a measure of the persistence of the autocorrelation of the Lagrangian velocity. A larger C_o determines a shorter decorrelation time scale T_L, which implies faster separation between the particles. When the respective velocity autocovariances of two particles are smaller, the particles travel together for shorter times as they tend to move independently of each other. Likewise, because the particle trajectories tend to be independent of each other, their centre of mass tends more rapidly toward stationarity.

The complete evolution in terms of relative and absolute dispersion that emerges from (4.6) is counterintuitive. As is well known, when T_L decreases because of larger values of the energy dissipation ε or of C_o, the cloud undergoes to overall slower dispersion as $\langle y^2 \rangle$ is smaller; however, at the same time, the cloud relative expansion $\langle y_r^2 \rangle$ is enhanced, causing a faster suppression of the meandering oscillations, and a faster decay of the internal scalar fluctuations. By contrast, note that several sensitivity tests to C_o that were performed in two-particle Lagrangian stochastic models (e.g. Borgas & Sawford 1994; Kurbanmuradov *et al.* 2001; Franzese & Borgas 2002) show that an increase in C_o causes a lower value of C_r, namely a slower separation, in conflict with (4.6). The reasons for this anomaly are explained in § 4.2, where a relation is established between second-order Eulerian velocity structure function and relative dispersion energy $\langle v_r^2 \rangle$. The results in § 4.2 also provide the means for validating the present theory with DNS and experimental data at finite Reynolds numbers.

4.2. *Eulerian velocity structure function and relative dispersion energy*

The relative dispersion energy $\langle v_r^2\rangle$ is used to establish a relation between the Kolmogorov constants in the Eulerian and Lagrangian velocity structure functions. We consider the Eulerian longitudinal velocity structure function $D_{LL}(d)=\langle[u_L(\boldsymbol{x}+\boldsymbol{d})-u_L(\boldsymbol{x})]^2\rangle$, where $\boldsymbol{x}$ is a fixed point in space, u_L is the component of the Eulerian velocity $\boldsymbol{u}$ parallel to the vector $\boldsymbol{d}$, and $d=|\boldsymbol{d}|$. For very high Reynolds number, an inertial subrange is defined for $1 \ll d/\eta \ll Re^{3/4}$, where the Kolmogorov microscale $\eta=(\nu^3/\varepsilon)^{1/4}$. In the inertial subrange, D_{LL} takes the form:

$$D_{LL}(d) = C_k(\varepsilon d)^{2/3}, \tag{4.7}$$

where C_k is the Kolmogorov constant. Equation (4.7) can be expressed as:

$$D_{LL}(d) = 2\langle u^2\rangle\left(\frac{d^2}{\mathscr{L}_E^2}\right)^{1/3}, \tag{4.8}$$

where

$$\mathscr{L}_E^2 = \left(\frac{2}{C_k}\right)^3\frac{\langle u^2\rangle^3}{\varepsilon^2}. \tag{4.9}$$

Simple approximate forms of (4.8) have been used often (Durbin 1980; Sawford & Hunt 1986; Thomson 1990) without an explicit dependence on C_k, by simply assuming $\mathscr{L}_E^2=\langle u^2\rangle^3/\varepsilon^2$ (in fact, implicitly assuming $C_k=2$).

Because $\mathscr{L}_E^2 \propto \langle u^2\rangle^3/\varepsilon^2$ and $\mathscr{L}_L^2 \propto \langle v^2\rangle^3/\varepsilon^2$, it follows that $\mathscr{L}_E^2 \propto \mathscr{L}_L^2$, which can be written in the form

$$\mathscr{L}_E^2 = 6C_\sigma\mathscr{L}_L^2 = 6C_\sigma C_L\frac{4\langle v^2\rangle^3}{C_o^2\varepsilon^2}, \tag{4.10}$$

where $\mathscr{L}_L^2$ was given by (4.2), and C_σ is a constant of proportionality. The factor 6 is unessential as it could be included in the constant C_σ. It is explicitly reported in (4.10) only because it accounts for the different magnitude between a two-particle length scale in three dimensions (such as $\mathscr{L}_E$), and a measure of distance from the centre of mass in one dimension (such as $\mathscr{L}_L$), as described by (2.9). From (4.9) and (4.10), we have:

$$C_k = \frac{C_o^{2/3}}{(3C_\sigma C_L)^{1/3}}. \tag{4.11}$$

A power-law relation between C_k and C_o is to be expected on physical grounds, as the result of a relation between integral temporal and spatial scales (Maurizi, Pagnini & Tampieri 2004). Equation (4.8) shows that the term $(d/\mathscr{L}_E)^{2/3}$ acts as a filter for the turbulent energy $\langle u^2\rangle$, i.e. D_{LL} is proportional to the fraction of $\langle u^2\rangle$ which is effective for the relative dispersion of particles separated by a distance d. Equation (4.10) shows that if C_o increases (e.g. as a consequence of an increase of the Reynolds number), $\mathscr{L}_E$ decreases. Therefore, according to (4.8), a larger fraction of the total energy is available for the separation at the fixed distance d. This corresponds to a larger C_k as shown by (4.7) and (4.11). Also, a higher rate of separation between marked particles, represented by a larger C_r, is expected, as shown by (4.6).

The anomalous inverse relation between C_o and C_r observed in stochastic Lagrangian models (Borgas & Sawford 1994; Kurbanmuradov *et al.* 2001; Franzese & Borgas 2002) arises from the violation of (4.11). Increasing C_o with a fixed C_k determines a spurious increase in C_σ, namely, the proportion between Eulerian and Lagrangian scales is altered, with an overestimated value of $\mathscr{L}_E$. In such conditions, the particles separate at a slower rate because the fraction of energy used for the separation process is underestimated.

Equation (4.11) will be tested against DNS data at various Reynolds numbers in § 8, after applying the correction for finite-Reynolds-number effects described in the next section.

4.3. *Reynolds number effects*

In this section, we will show that the relations between the classical turbulence constants C_r, C_o and C_k established in the previous sections imply specific Reynolds number effects on relative dispersion. The relations derived in this section will be used in § 8 for the comparisons with DNS and experimental data which are available at low to medium Reynolds number. The dependence of C_r and C_k on the Reynolds number is determined on the basis of the effects of the Reynolds number on C_o. Sawford (1991) used a second-order autoregressive Lagrangian model for acceleration of fluid particles to show that the Lagrangian time scale T_L depends on the Reynolds number, with high sensitivity for $Re_\lambda \lesssim 10^3$, where $Re_\lambda = \sqrt{\langle u^2 \rangle}\lambda/\nu$ is the Reynolds number based on the Taylor microscale $\lambda = \sqrt{15\langle u^2 \rangle \nu/\varepsilon}$, which gives $Re_\lambda = \sqrt{15 Re}$. As a consequence, Sawford (1991) found that the effective value of C_o determined from the Reynolds-number-dependent T_L is approximated well by

$$\widetilde{C}_o = C_o f(Re_\lambda), \tag{4.12}$$

where $\widetilde{C}_o$ is the value of C_o at finite Reynolds number, and f is an asymptotic function which increases with Re_λ, and tends to 1 as Re_λ tends to infinity.

From (4.6) and (4.12), we find that the explicit dependence of C_r on Re_λ is simply written as:

$$\widetilde{C}_r = C_r f(Re_\lambda), \tag{4.13}$$

where $\widetilde{C}_r$ is the value of C_r at finite Reynolds number, and C_r is given by (4.4). The Reynolds number effects on C_k are obtained using (4.12) along with (4.11):

$$\widetilde{C}_k = C_k f(Re_\lambda)^{2/3}, \tag{4.14}$$

where $\widetilde{C}_k$ is the value of C_k at finite Reynolds number.

Finally, the explicit dependence of $\mathscr{L}_E$ on Re_λ can be obtained using (4.12):

$$\widetilde{\mathscr{L}}_E = \mathscr{L}_E f(Re_\lambda)^{-1}, \tag{4.15}$$

where the inverse relation between $\widetilde{\mathscr{L}}_E$ and Re_λ shows that, at finite Re_λ, the velocities are correlated at larger distances than for the asymptotic large Re_λ limit, consistently with the description given in § 4.2.

We will use Sawford's (1991) estimate for f:

$$f(Re_\lambda) = \left(1 + 7.5 C_o^2 Re_\lambda^{-1.64}\right)^{-1}, \tag{4.16}$$

where C_o was estimated to be about 7 based on the comparisons with DNS data. This is also the value we will use in our comparisons. Sawford's model with (4.16) accurately reproduces the longitudinal Lagrangian velocity structure function $\langle (\Delta_\tau v)^2 \rangle$ obtained by the DNS of Yeung & Pope (1989), for Re_λ ranging from about 40 to about 90. It is possible to use other approximations for f: for example, Lien & D'Asaro (2002) proposed $f = 1 - \sqrt{10/Re_\lambda}$, which gives results comparable to those obtained using (4.16).

Equation (4.12) was tested against DNS data by Sawford (1991); equations (4.13) and (4.14) will be used in the tests against DNS data and a laboratory experiment in § 8.

5. Large-scale solution

A complete analytical formulation of $\langle y_r^2 \rangle$, beyond inertial subrange scales, can be derived using (4.1) for $\langle y_r^2 \rangle \leqslant \mathscr{L}_L^2$, and $\langle v_r^2 \rangle = \langle v^2 \rangle$ for $\langle y_r^2 \rangle > \mathscr{L}_L^2$. This is a somewhat crude approximation, which has the advantage of providing a simple formulation for the relative dispersion spanning all scales of motion. The results are valid at inertial subrange time scales and at large times, but are only approximated in the transition region $\langle y_r^2 \rangle \simeq \mathscr{L}_L^2$. In this region, an accurate estimate of $\langle v_r^2 \rangle$ is required in order to reproduce the details of dispersion.

Because the solution to (3.4) for $\langle y_r^2 \rangle \leqslant \mathscr{L}_L^2$ is $\langle y_r^2 \rangle = C_y \varepsilon t^3$, the time T_y at which $\langle y_r^2 \rangle = \mathscr{L}_L^2$ can be approximated by:

$$C_y \varepsilon T_y^3 \approx \mathscr{L}_L^2, \tag{5.1}$$

which defines $T_y = \beta T_L$, with $\beta = (3\alpha/4)^{-1/3}$. Therefore, the solution to (3.2) is:

$$\left\langle y_r^2 \right\rangle = C_y \varepsilon t^3 \qquad \text{for } t \leqslant T_y, \tag{5.2a}$$

$$\left\langle y_r^2 \right\rangle = \mathscr{L}_L^2 + 2\int_{T_y}^{t} \left\langle v_r^2 \right\rangle T_r \mathrm{d}t = \mathscr{L}_L^2 + 2\langle v^2 \rangle T_L (t - T_y) \qquad \text{for } t \geqslant T_y. \tag{5.2b}$$

Equations (5.2a) and (5.2b) can be rewritten as:

$$\left\langle y_r^2 \right\rangle = 2\langle v^2 \rangle T_L^2 \alpha \left(\frac{t}{T_L}\right)^3 \qquad \text{for } t \leqslant T_y, \tag{5.3a}$$

$$\left\langle y_r^2 \right\rangle = 2\langle v^2 \rangle T_L^2 \left(\frac{C_L}{2} - \beta + \frac{t}{T_L}\right) \qquad \text{for } t \geqslant T_y, \tag{5.3b}$$

which emphasize the analogy with the absolute dispersion theory: $\langle y_r^2 \rangle$ depends on the same turbulence variables as $\langle y^2 \rangle$, namely kinetic energy and Lagrangian time scale. As a consequence, the representation of the non-dimensional variable $\langle y_r^2 \rangle / (\langle v^2 \rangle T_L^2)$ as a function of t/T_L is unique, consistent with the unique representation of $\langle y^2 \rangle / (\langle v^2 \rangle T_L^2)$ as a function of t/T_L.

The consistency with Taylor's formulation for $\langle y^2 \rangle$ is a consequence of the application of the statistical diffusion theory to relative dispersion, along with the definition of $\langle v_r^2 \rangle$ in (4.1) as a function of the same Lagrangian quantities employed in the derivation of $\langle y^2 \rangle$. As is to be expected on physical grounds, every change in the flow characteristics that affects $\langle y^2 \rangle$ (through $\langle v^2 \rangle$ or T_L) has a corresponding effect on $\langle y_r^2 \rangle$, while still satisfying the realizability condition $\langle y_r^2 \rangle < \langle y^2 \rangle$, which is not immediately apparent from the classical form $\langle y_r^2 \rangle = C_y \varepsilon t^3$.

6. Finite source size

We consider now the case of a cluster with an initial mean square particle separation, corresponding to a source with a finite size. In the vicinity of a finite source, the t^3 inertial subrange scaling for $\langle y_r^2 \rangle$ breaks down because the statistical properties of v_r at the source are affected by the initial conditions. A clear illustration of the source effects is provided by the average rate of expansion of a particle pair, which is zero at the source (because at $t=0$ it is equivalent to an Eulerian statistic), but is positive at any time after release (e.g. Faller 1996).

Batchelor (1952) carried out an analysis of source effects on the relative dispersion of pairs of particles initially separated by a distance $r_o=|\boldsymbol{r}_o|$. If $1 \ll r_o/\eta \ll Re^{3/4}$, the source effects are short time ranged and last for an initial time lapse $t \lesssim \tau_s=(r_o^2/\varepsilon)^{1/3}$, namely the effects of initial conditions rapidly vanish as the evolution of $\langle \boldsymbol{r}^2\rangle$ tends to be uninfluenced by the source, and to conform to a t^3 behaviour. It is possible to extend Batchelor's formulation to clusters of particles, but the results are not general because they depend on the specific distribution function at the source (see the Appendix). In any case, the complete evolution from near-source behaviour to intermediate asymptotic scaling law in the inertial subrange is not known, and is usually modelled using an approximation for the source distribution and a heuristic interpolation between the two regimes (Batchelor 1950; Luhar, Hibberd & Borgas 2000).

We use a simple approximate formulation for expansion from finite sources which has some advantages: this approach naturally ensures consistency with the formulation at larger times avoiding the need for empirical interpolations at $t \sim \tau_s$ and the introduction of new parameters; the resulting equations have the desirable properties of representing expansion and rate of expansion as continuous and differentiable processes (Franzese 2003; Goto & Vassilicos 2004; Cassiani, Franzese & Giostra 2005).

The formulation is obtained enforcing a principle of self-similar expansion for releases from different source sizes, namely, the evolution of $\langle y_r^2\rangle$ at a given time t does not depend on $\langle y_r^2(0)\rangle$, but only on the instantaneous value of $\langle y_r^2\rangle$. For sources in the inertial subrange, self-similar relative expansion is naturally derived directly from the original Richardson 4/3 law, which can be written in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle y_r^2\right\rangle \propto \left\langle y_r^2\right\rangle^{2/3} = 3(C_y\varepsilon)^{1/3}\left\langle y_r^2\right\rangle^{2/3}. \tag{6.1}$$

Integration of (6.1) gives:

$$\left\langle y_r^2\right\rangle = C_y\varepsilon(t_s+t)^3, \tag{6.2}$$

where $t_s=[\langle y_r^2(0)\rangle/(C_y\varepsilon)]^{1/3}$. In fact, t_s corresponds physically to the time required for a virtual point source release to expand to the mean square size $\langle y_r^2(0)\rangle$.

In general, self-similarity can be simply applied by imposing a translation of the time origin by an amount t_s in the equations of $\langle y_r^2\rangle$ for a point source. Therefore, the time-dependent decorrelation time scale T_r for finite sources is expressed as:

$$T_r = \left[\frac{C_o\varepsilon}{2\left\langle v_r^2\right\rangle} + \frac{1}{2(t_s+t)}\right]^{-1} \tag{6.3}$$

and the complete solution is:

$$\left\langle y_r^2\right\rangle = 2\langle v^2\rangle T_L^2\alpha\left(\frac{t_s+t}{T_L}\right)^3 \qquad \text{for } t \leqslant T_y - t_s, \tag{6.4a}$$

$$\left\langle y_r^2\right\rangle = 2\langle v^2\rangle T_L^2\left(\frac{C_L}{2}-\beta+\frac{t_s+t}{T_L}\right) \qquad \text{for } t \geqslant T_y - t_s. \tag{6.4b}$$

Note that (6.4*a*) and (6.4*b*) depend on the same variables as (5.3*a*) and (5.3*b*). In § 8, these formulae will be represented along with $\langle y^2\rangle$ to give a complete picture of relative and absolute dispersion from the source until the diffusive regime.

7. A closure for the relative turbulent kinetic energy $\langle v_r^2 \rangle$

The equations for $\langle y_r^2 \rangle$ derived in the previous sections are unclosed in that the large-eddy length scale $\mathscr{L}_L$, which appears in the formulation of $\langle v_r^2 \rangle$ in (4.1), is not known. In this section, we propose a simple and physical closure assumption for $\mathscr{L}_L$ consistent with $\langle y_r^2 \rangle$, which was used as a length scale for the cluster size in the definition of $\langle v_r^2 \rangle$ given in (4.1). This will also permit quantitative comparisons with DNS and experimental data, including a theoretical prediction for the Richardson–Obukhov constant C_r.

$\mathscr{L}_L^2$ corresponds to the value of $\langle y_r^2 \rangle$ when the cloud has reached a size comparable to the large-eddy length scale. A sound estimate of $\mathscr{L}_L^2$ is given by $\langle y^2 \rangle$ when it has reached a value comparable to the large-eddy length scale, and starts evolving according to the Brownian diffusion regime, i.e. $\mathscr{L}_L^2 = \langle y^2 \rangle$ at the onset of the diffusion regime.

The problem is that the transition of $\langle y^2 \rangle$ to the diffusion regime is not sharp and unequivocal, because the autocorrelation of Lagrangian velocity $R(\tau) = \langle v(t)v(t+\tau)\rangle / \langle v^2 \rangle$ is asymptotically decaying, with smooth regime transitions. In order to identify as accurately as possible a virtual transition time to Brownian diffusion, we will use a linear form for $R(\tau)$ (Pasquill 1962, p. 92; Monin & Yaglom 1971, p. 578; Arya 1999, p. 157), which exactly defines a transition time to perfectly uncorrelated velocities. The value of $\langle y^2 \rangle$ corresponding to this time defines $\mathscr{L}_L^2$.

$R(\tau)$ is defined as

$$R(\tau) = 1 - \frac{\tau}{2T_L} \qquad \text{for } \tau \leqslant 2T_L, \tag{7.1a}$$

$$R(\tau) = 0 \qquad \text{for } \tau \geqslant 2T_L, \tag{7.1b}$$

which formally define the onset of a Brownian diffusion regime after a dispersion time $t = 2T_L$. Application of Taylor's (1921) theory for a point source gives:

$$\langle y^2 \rangle = \langle v^2 \rangle T_L^2 \left(\frac{t^2}{T_L^2} - \frac{t^3}{6T_L^3} \right) \qquad \text{for } t \leqslant 2T_L, \tag{7.2a}$$

$$\langle y^2 \rangle = \langle v^2 \rangle T_L^2 \left(\frac{2t}{T_L} - \frac{4}{3} \right) \qquad \text{for } t \geqslant 2T_L, \tag{7.2b}$$

which provide the value of $\mathscr{L}_L^2$, defined in (4.2):

$$\mathscr{L}_L^2 = \langle y^2(2T_L) \rangle = C_L \langle v^2 \rangle T_L^2 \quad \text{with} \quad C_L = \tfrac{8}{3}. \tag{7.3}$$

The proportionality constant $\sqrt{C_L}$ establishes an exact correspondence between the length scale of the energy-containing vortices as expressed by $\langle v^2 \rangle^{3/2}/\varepsilon$, and a specific measure of their size obtained by calculating the standard deviation of a (Gaussian) distribution of marked fluid particles, $\mathscr{L}_L$. Equations (7.2a) and (7.2b) are in general a good approximation to the solution obtained for the more common exponential form $R(\tau) = \exp(-|\tau|/T_L)$, and display the prescribed well-known features of $\langle y^2 \rangle$ at small and large travel times: $\langle y^2 \rangle \simeq \langle v^2 \rangle t^2$ for $t \ll T_L$ and the Brownian diffusion $\langle y^2 \rangle \simeq 2\langle v^2 \rangle T_L t$ for $t \gg T_L$.

Figure 2 shows the evolution of $\langle y^2 \rangle$ as described by (7.2a) and (7.2b), normalized by $\langle v^2 \rangle T_L^2$, as a function of the normalized dispersion time t/T_L. The squared non-dimensional length scale $\mathscr{L}_L^2 / \left(\langle v^2 \rangle T_L^2 \right)$ is also displayed in the figure, emphasizing the relatively sharp transition from the $\langle y^2 \rangle \sim t^2$ regime to the $\langle y^2 \rangle \sim t$ regime. The internal frame displays an enlargement of the transition region, and includes the additional plot of $\langle y^2 \rangle$ for exponential $R(\tau)$, i.e. $\langle y^2 \rangle = 2\langle v^2 \rangle T_L^2 \left[t/T_L + \exp(-t/T_L) - 1 \right]$

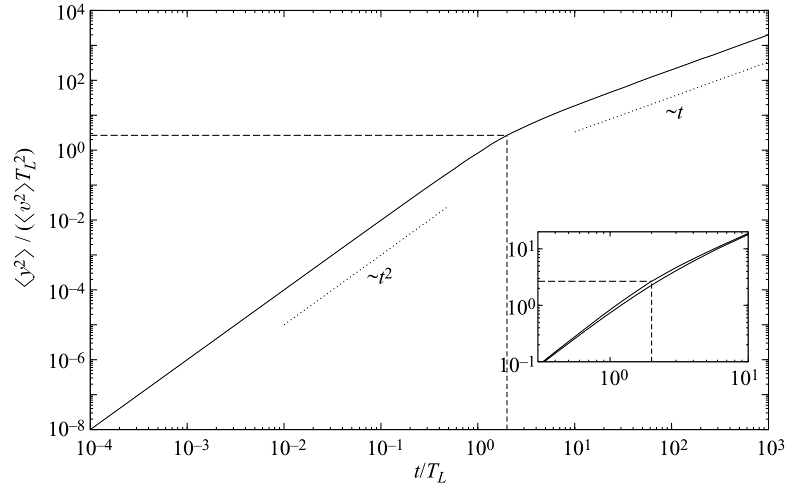

FIGURE 2. The evolution of the absolute dispersion variance $\langle y^2\rangle$ in (7.2*a*) and (7.2*b*) for the linear autocorrelation function (7.1*a*) and (7.1*b*). The horizontal dashed line corresponds to the square of the non-dimensional large-eddy length scale $\mathscr{L}_L$. The enlargement in the internal frame includes the additional plot of $\langle y^2\rangle$ for exponential $R(\tau)$, lying slightly below the original curve. The straight lines are proportional to t and t^2.

lying slightly below the original curve, within a maximum relative error of less than 15%. It is clear from figure 2 that the exact form of $R(\tau)$ is relatively unimportant in the determination of the dispersion parameters. In fact, various alternative forms have been observed not to produce fundamental differences in the dispersion characteristics, as long as the expressions of $R(\tau)$ generate the same Lagrangian time scale $T_L = \int_0^\infty R(\tau)\mathrm{d}\tau$ (Pasquill 1962, p. 92; Arya 1999, p. 157).

8. Comparisons with experiments and DNS

It is now possible to test with experimental and computational data some of the quantities and relations predicted by the present theory.

The Richardson–Obukhov constant C_r in (4.6) is obtained substituting (7.3) into (4.5), which then gives $\alpha = (9\sqrt{6} - 22)/3$. This value can also be extracted directly from a numerical solution of (3.18). Therefore, we have:

$$C_r = (18\sqrt{6} - 44)C_o, \tag{8.1}$$

or $C_r \approx C_o/11$. Assuming $C_o = 7$, we obtain $C_r = 0.64$.

To test (8.1), the predicted C_r is used in (4.13), which provides C_r as a function of the Taylor-scale Reynolds number Re_λ, namely $\widetilde{C}_r$. Equation (4.13) is plotted in figure 3 along with the data from Ott & Mann's (2000) experiment and from the DNS of Ishihara & Kaneda (2002), Biferale *et al.* (2005) and Boffetta & Sokolov (2002*a*).

Ott & Mann (2000) detected inertial subrange scaling for relative dispersion in low Re_λ ($\simeq 100$) turbulence generated by oscillating grids. The values of the measured $\widetilde{C}_r$ in Ott & Mann's dispersion experiments indicate an average of $\widetilde{C}_r \simeq 0.5$ with an estimated uncertainty of about 10%. The calculations were based on the assumption that the Kolmogorov constant $C_k = 2$. The DNS of Ishihara & Kaneda (2002) at $Re_\lambda = 283$ also show inertial subrange scaling of relative dispersion for the releases

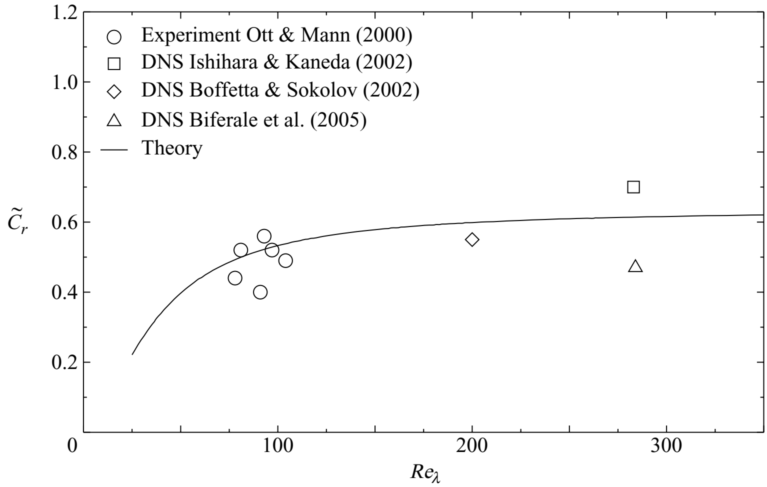

FIGURE 3. The relation between the Richardson–Obukhov constant $\widetilde{C}_r$ and Re_λ predicted by (4.13) (solid line), along with laboratory and DNS data (symbols).

with initial separation $r_o = 5\eta$. The estimated $\widetilde{C}_r$ increases with the initial separation and converges to a value $\widetilde{C}_r \simeq 0.7$ for all $r_o = 20\eta$. The curves generated by the DNS of Biferale *et al.* (2005) at $Re_\lambda = 284$ do not display a clear t^3 scaling and show a dependence on the initial separation. However, the authors use the same extrapolation procedure as in Ott & Mann (2000) and Ishihara & Kaneda (2002) to estimate $\widetilde{C}_r = 0.47$ with a reported error of the order of approximately 10%. In the DNS of Boffetta & Sokolov (2002*a*) at $Re_\lambda \simeq 200$, doubling-time analysis was used to estimate $\widetilde{C}_r \simeq 0.55$.

Although the value $C_r = 0.64$ predicted by the present theory at high Reynolds number is supported by the reported data, it is worth emphasizing that experimental and numerical estimates of C_r are, at present, still uncertain, and a definitive conclusion on the best estimate of C_r cannot be drawn from the available data.

We plot in figure 4 the complete solution for $\langle y_r^2 \rangle$ in (6.4*a*) and (6.4*b*) for a source $\langle y^2(0) \rangle = 10^{-9} \langle v^2 \rangle T_L^2$, along with Taylor's $\langle y^2 \rangle$ obtained for an exponential autocorrelation function. The mean square variables are normalized over $\langle v^2 \rangle T_L^2$. It is emphasized that because $\langle y_r^2 \rangle$ was determined entirely in terms of Lagrangian variables (i.e. $\langle v^2 \rangle$ and T_L) the variable $\langle y_r^2 \rangle / (\langle v^2 \rangle T_L^2)$, displayed in figure 4, has a unique representation as a function of t/T_L, likewise $\langle y^2 \rangle$. The figure shows the smooth transition between scaling regimes, and the consistency between $\langle y^2 \rangle$ and $\langle y_r^2 \rangle$ at the asymptotic limits. In particular, the large-time behaviour of $\langle y_r^2 \rangle$ is not a consequence of *ad hoc* assumptions, but is a natural result of the theory.

The Kolmogorov constant C_k provided by (4.11) is also tested with the results of DNS of homogeneous turbulence at several Reynolds numbers reported in the literature. For the comparison we use (4.14), which is based on (4.11) and describes C_k as a function of Re_λ, namely $\widetilde{C}_k$. Although there is some degree of uncertainty in the asymptotic value of $\widetilde{C}_k$, there seems to be a certain consensus on $C_k = 2.13$, which is the estimate proposed by Sreenivasan (1995) as a result of a comprehensive literature review of experiments and numerical simulations. This is also the value we will use in our comparisons.

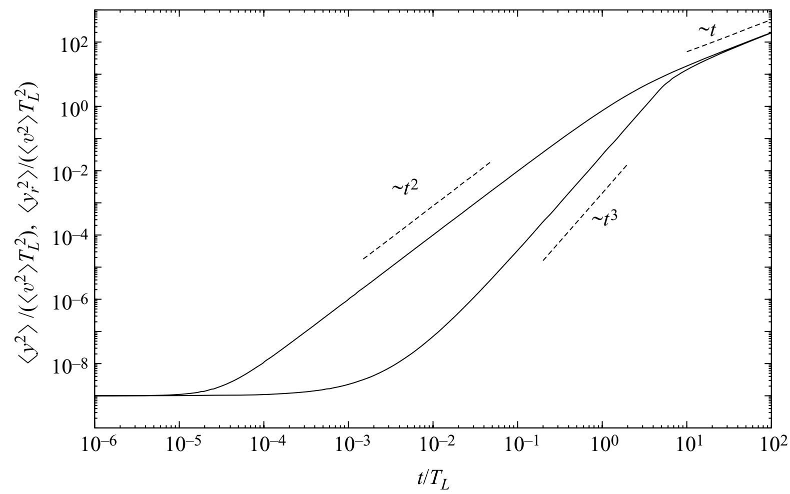

FIGURE 4. The evolutions of Taylor's absolute dispersion variance $\langle y^2\rangle$, and relative dispersion variance $\langle y_r^2\rangle$ [(6.4*a*) and (6.4*b*)] for an initial source $\langle y^2(0)\rangle = 10^{-9}\langle v^2\rangle T_L^2$. The straight lines are proportional to t, t^2 and t^3.

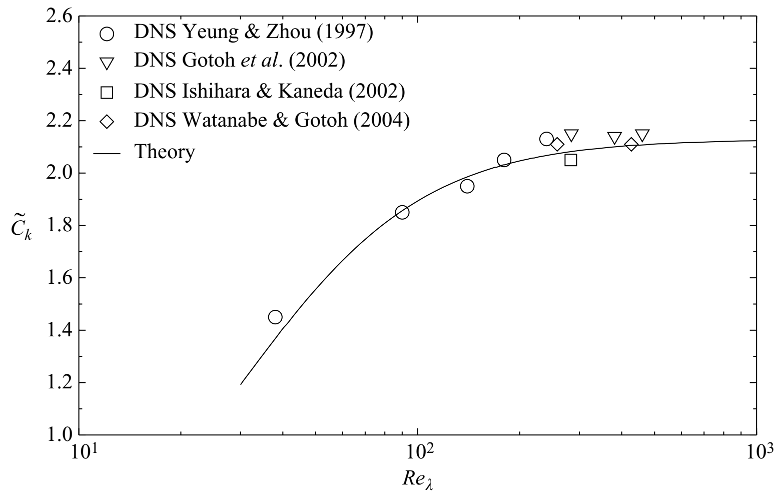

FIGURE 5. The relation between the Reynolds-dependent Kolmogorov constant for the Eulerian velocity structure function $\widetilde{C}_k$ and Re_λ predicted by (4.14) (solid line), along with data from various DNS (symbols).

Figure 5 shows values of $\widetilde{C}_k$ obtained by the DNS at several Reynolds numbers performed by Yeung & Zhou (1997), Ishihara & Kaneda (2002), Gotoh, Fukayama & Nakano (2002), and Watanabe & Gotoh (2004) along with (4.14) plotted as a continuous line. The data from Yeung & Zhou have been estimated as the maxima of the compensated Eulerian second-order longitudinal velocity structure function, i.e. $D_{LL}(d)/(\varepsilon d)^{2/3}$. It should be noted that only an approximated inertial subrange behaviour can be inferred from Yeung & Zhou's data, because the structure functions do not show a well-defined plateau at $Re_\lambda \leqslant 240$. In all other cases, $\widetilde{C}_k$ was calculated

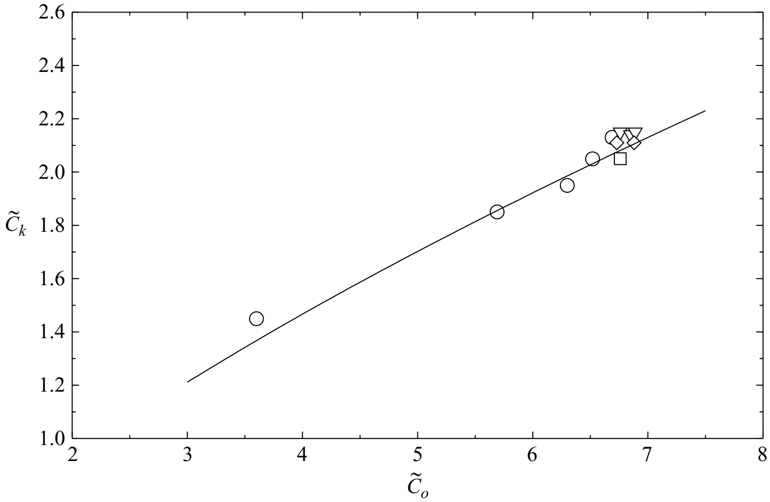

FIGURE 6. The relation between the Reynolds-dependent Kolmogorov constants for the Eulerian ($\widetilde{C}_k$) and Lagrangian ($\widetilde{C}_o$) velocity structure functions predicted by (4.11) (solid line), along with data from various DNS (symbols). The DNS data are plotted as functions of $\widetilde{C}_o$ using Sawford's relation (4.12). Key as for figure 5.

from the original estimated values of the Kolmogorov constant in the energy spectra provided by the respective authors.

The same finite-Reynolds number DNS data for $\widetilde{C}_k$ reported in figure 5 have been plotted in figure 6 as functions of $\widetilde{C}_o$ as calculated by (4.12). Equation (4.11) is also displayed in figure 6. The data clearly support the relation $C_k \propto C_o^{2/3}$ established by (4.11), and are in good quantitative agreement with the theory.

Figure 7 shows the predicted $R_r(t, \tau) = \exp(-\tau/T_r)$ as a function of τ/t. According to (3.17) and (4.6), T_r in the inertial subrange can be expressed as:

$$T_r = \left(\frac{2}{1 + 4(6\alpha)^{-1/3}} \right) t. \tag{8.2}$$

The linear dependence of T_r on t implies that $R_r(t, \tau)$ is a function of the single variable τ/t. The experimental data for $R_r(t, \tau)$ as a function of τ/t in inverse energy cascade two-dimensional turbulence reported in Jullien *et al.* (1999) along with the DNS data of Biferale *et al.* (2005) in three-dimensional turbulence are also plotted in figure 7. The function R_r is expected to be qualitatively and quantitatively similar in two- and three-dimensional turbulence. Kolmogorov $k^{-5/3}$ and Richardson–Obukhov t^3 scaling have been observed numerically and experimentally within the inverse energy cascade in two-dimensional turbulence, with different C_k, C_r and C_o from those which are observed in three-dimensional turbulence (Babiano *et al.* 1990; Jullien *et al.* 1999; Boffetta & Sokolov 2002*b*; Goto & Vassilicos 2004). Our expression (8.2) for T_r shows that R_r is based on the ratio C_r/C_o, and should therefore be independent of dimensionality. The data in figure 7 seem to confirm this feature. Similar behaviour has been observed in the DNS of Goto & Vassilicos (2004) for a slightly different definition of R_r.

The experimental data in figure 7 are in reasonably good quantitative agreement with the prediction. However, it is emphasized that all data clearly support the

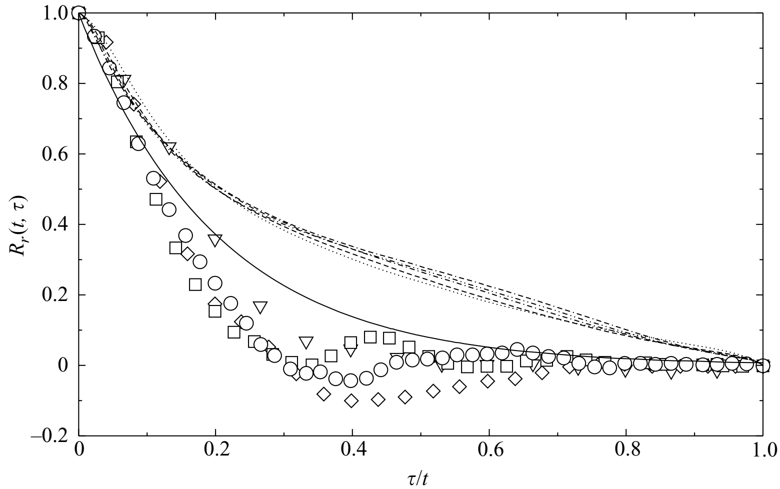

FIGURE 7. The relative velocity autocorrelation function $R_r(t, \tau)$ as a function of the single variable τ/t predicted by the theory (solid line), along with laboratory data of Jullien *et al.* (1999) (symbols) and the DNS data of Biferale *et al.* (2005) (broken lines), recorded at various times after release. ∇, $t = 3s$; $\diamond$, $t = 5s$; $\square$, $t = 7s$; $\circ$, $t = 9s$; dotted line, $t = 21\tau_\eta$; dashed line, $t = 35\tau_\eta$; dot-dash line, $t = 49\tau_\eta$; dot-dot-dash line, $t = 63\tau_\eta$; dot-dash-dash line, $t = 77\tau_\eta$.

predicted dependence of $R_r(t, \tau)$ on the single variable τ/t, which is an important confirmation of the theory.

9. Discussion

The derivation of a differential equation for the mean square relative separation based on first principles, with a physically reasonable form of the kinetic energy used by the separation motions, is attractive because it encompasses several constraints, chiefly the consistency with Taylor's absolute dispersion theory. The relative dispersion theory formulated in this paper is based on purely statistical and kinematical considerations, and satisfies by construction the constraint $\langle y_r^2 \rangle < \langle y^2 \rangle$ at all times, which is especially important when parameterized forms of $\langle y_r^2 \rangle$ are used in concentration fluctuation analyses, or to isolate the contribution of internal fluctuations (Wilson 1995; Luhar *et al.* 2000; Cassiani & Giostra 2002; Franzese 2003). The present theory is also instrumental in identifying quantities specific to relative dispersion, such as relative velocity autocorrelation function R_r, relative dispersion time scale T_r, and relative separation energy $\langle v_r^2 \rangle$. The simplicity of the approach allows for a detailed analysis of relative dispersion dynamics including the complementary process of the meandering of the centre of mass.

Several studies show that $\langle y^2 \rangle$ is not as sensitive to the form of the velocity autocovariance as to the Lagrangian decorrelation time scale. We find that, in the case of relative dispersion, this property is more pronounced: the form of R_r is in effect irrelevant, as the evolution equation can be written directly in terms of the time integral of R_r, which defines the relative dispersion time scale T_r. Time scales of relative dispersion have been related to the characteristic time scale of dissipation of scalar fluctuations (e.g. Sykes, Lewellen & Parker 1984; Sawford 2004; Cassiani *et al.* 2005), which is used to define the micromixing time in PDF micromixing models such as the IEM model (interaction by exchange with the mean, Villermaux & Devillon

1972), and the IECM model (interaction by exchange with the conditional mean, Fox 1996).

The results obtained in this paper are extended to different Reynolds numbers using relations based on Sawford's (1991) definition of a Lagrangian time scale dependent on Reynolds number, which proved to be accurate in the original comparisons with the DNS of Yeung & Pope (1989), and in more recent analyses of the Kolmogorov constant for the Lagrangian velocity spectrum (Lien & D'Asaro 2002). For instance, at $Re_\lambda \sim 75$, we find a variation of C_r with respect to its asymptotic (i.e. infinite Reynolds number) value of about 24%, and a variation of C_k of about 16%. At $Re_\lambda \lesssim 75$, the meaningfulness of estimates of C_r and C_k is questionable as Kolmogorov $k^{-5/3}$ scaling and Obukhov's law itself may not hold.

The uncertainty and variability of the computational and experimental data used in the paper should caution against final conclusions with regard to a reliable estimate of the Richardson–Obukhov constant C_r. As shown in this paper, estimates of other inertial subrange quantities such as the Lagrangian and Eulerian structure function Kolmogorov constants C_o and C_k should be an integral part of a final assessment of C_r because of the interrelations between these constants. Additional relations between C_r and the Corrsin–Obukhov constant C_ϑ were found by Larchevêque & Lesieur (1981) (see also Thomson 1996).

We thank David Thomson and Brian Sawford for their comments and encouragement, the supercomputing centre CINECA (Bologna, Italy) for the hosting of the DNS data on the relative velocity autocorrelation function and Alessandra Lanotte for her kind assistance with these data. It is a pleasure to express here our gratitude to Rainald Löhner for his unconditional support.

Appendix. Finite source

Taylor's formula is based on the assumption that all particles are released from the same point. If the source has a finite size, it can be regarded as an ensemble of point sources distributed according to the source shape. In this case, $\langle y^2\rangle$ is defined by the average over the ensemble of realizations, and over all point sources which define the actual source. In general, the two-time displacement autocovariance over the ensemble of point sources can be written as:

$$\overline{[y(t_1) - y(0)][y(t_2) - y(0)]} = \int_0^{t_1}\int_0^{t_2} \overline{v(t')v(t'')}\,\mathrm{d}t'\,\mathrm{d}t'', \qquad (\mathrm{A}\,1)$$

where the independence of $\overline{v(t')v(t'')}$ from $y(0)$ in homogeneous turbulence was used, and $t_o = 0$ was assumed. Therefore

$$\langle y^2(t)\rangle = \langle y^2(0)\rangle + \int_0^{t}\int_0^{t} \langle v(t')v(t'')\rangle\,\mathrm{d}t'\,\mathrm{d}t'', \qquad (\mathrm{A}\,2)$$

and Taylor's formula for a finite source size $\sqrt{\langle y^2(0)\rangle}$ is simply written as:

$$\langle y^2(t)\rangle = \langle y^2(0)\rangle + 2\langle v^2\rangle \int_0^{t} (t-\tau)R(\tau)\,\mathrm{d}\tau. \qquad (\mathrm{A}\,3)$$

The effects of a finite source on relative dispersion are less straightforward. Batchelor's (1952) analysis of source effects on the relative dispersion of pairs of particles initially separated by a distance $r_o = |\boldsymbol{r}_o|$ shows that, for sources in the

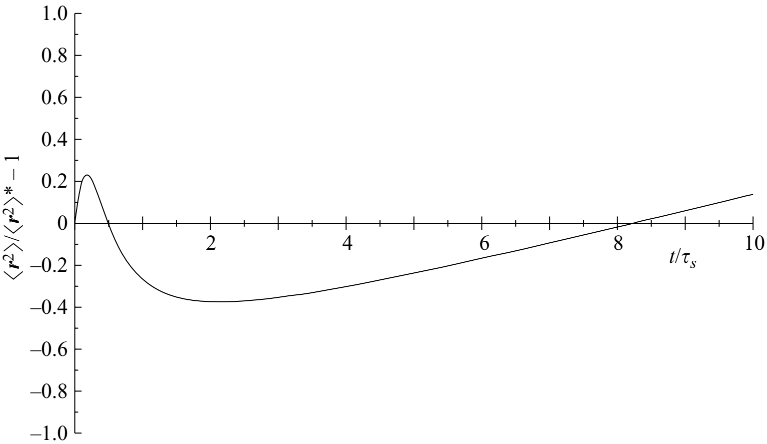

FIGURE 8. Relative error of two-particle mean square separation $\langle \boldsymbol{r}^2\rangle$, (A 8), with respect to Batchelor's $\langle \boldsymbol{r}^2\rangle^*$, (A 7), namely $\langle \boldsymbol{r}^2\rangle/\langle \boldsymbol{r}^2\rangle^* - 1$ as a function of t/τ_s.

inertial subrange, $\langle \boldsymbol{r}^2\rangle - r_o^2 \propto t^2$ for an initial time lapse $t \lesssim \tau_s = (r_o^2/\varepsilon)^{1/3}$:

$$\langle \boldsymbol{r}^2\rangle^* \approx r_o^2 + \langle[\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}_o) - \boldsymbol{u}(\boldsymbol{x})]^2\rangle t^2, \tag{A 4}$$

where $\langle \boldsymbol{r}^2\rangle^*$ indicates the approximate expression for $\langle \boldsymbol{r}^2\rangle$ according to Batchelor. Because of isotropy

$$\begin{aligned}\langle[\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}_o) - \boldsymbol{u}(\boldsymbol{x})]^2\rangle &= \langle[u_L(\boldsymbol{x}+\boldsymbol{r}_o) - u_L(\boldsymbol{x})]^2\rangle + 2\langle[u_N(\boldsymbol{x}+\boldsymbol{r}_o) - u_N(\boldsymbol{x})]^2\rangle \\ &= \tfrac{11}{3}C_k(\varepsilon r_o)^{2/3}\end{aligned} \tag{A 5}$$

where u_L and u_N are the velocity components parallel and perpendicular to the separation vector $\boldsymbol{r}_o$, respectively. For clusters of particles with a given initial distribution $p(r_o)$, further averaging over r_o is required, and we have:

$$\langle \boldsymbol{r}^2\rangle^* = \left\langle r_o^2\right\rangle + \tfrac{11}{3}C_k\varepsilon^{2/3}\left\langle r_o^{2/3}\right\rangle t^2. \tag{A 6}$$

If the particle pairs are released with the same initial separation, i.e. $p(r_o) = \delta(r - r_o)$, then (A 6) simplifies to

$$\langle \boldsymbol{r}^2\rangle^* = r_o^2\left[1 + \tfrac{11}{3}C_k(t/\tau_s)^2\right]. \tag{A 7}$$

We will use this expression to estimate the error that is made neglecting the source effect when using the approximate expression derived in § 6, for two-particle three-dimensional separation, with the same initial conditions, i.e.

$$\left\langle \boldsymbol{r}^2\right\rangle = C_r\varepsilon(t_s + t)^3, \tag{A 8}$$

where $t_s = [r_o^2/(C_r\varepsilon)]^{1/3}$.

The relative error is given by:

$$\frac{\langle \boldsymbol{r}^2\rangle - \langle \boldsymbol{r}^2\rangle^*}{\langle \boldsymbol{r}^2\rangle^*} = \frac{\left(1 + C_r^{1/3}t/\tau_s\right)^3}{1 + \frac{11}{3}C_k(t/\tau_s)^2} - 1. \tag{A 9}$$

A measure of maximum relative error is somewhat ambiguous, because (A 4) is valid only in the limit as t tends to zero, and a formula describing the transition to larger times is not known. However, figure 8 shows that the relative error, (A 9), plotted as a function of t/τ_s for $C_r = 0.64$ and $C_k = 2.13$, does not exceed 27% for $t \leqslant \tau_s$. The

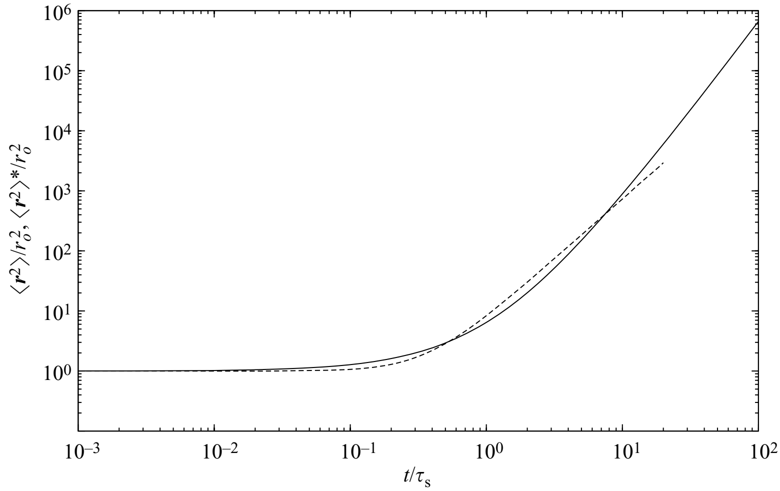

FIGURE 9. Near-source evolution of $\langle \boldsymbol{r}^2\rangle/r_o^2$ in (A 8) (solid line), along with $\langle \boldsymbol{r}^2\rangle^*/r_o^2$ in (A 7) (dashed line), as functions of t/τ_s.

difference at times larger than τ_s is less meaningful as the definition of $\langle \boldsymbol{r}^2\rangle^*$ lacks validity.

Figure 9 shows a detail of the small-time evolution of $\langle \boldsymbol{r}^2\rangle^*$ and $\langle \boldsymbol{r}^2\rangle$, plotted as functions of t/τ_s. The figure corroborates the assumption that the correct velocity initial conditions (which are accounted for in $\langle \boldsymbol{r}^2\rangle^*$) have, in fact, a marginal effect and do not generate fundamental differences from the behaviour prescribed by $\langle \boldsymbol{r}^2\rangle$. However, the importance of the comparison lies in the conclusion that the variation of the slope of $\langle \boldsymbol{r}^2\rangle$ for $t \lesssim 10\tau_s$ does not imply quadratic or non-inertial subrange scaling, but is simply a graphic effect of the representation of a non-zero initial $\langle \boldsymbol{r}^2\rangle$ on a log scale. In this respect, near-source deviations from the constant slope line that appear in experiments (e.g. Ott & Mann 2000) or in numerical simulations (e.g. Boffetta & Sokolov 2002*a*; Ishihara & Kaneda 2002; Gioia *et al.* 2004), while certainly symptomatic of a finite source size, may not be necessarily attributable to a quadratic (as opposed to cubic) behaviour in the entire time interval before a constant slope is reached; near-source inertial subrange scaling does not appear in the form of constant slope on a log–log plot if the source is finite.

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